Exponential stability of time-delay systems

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Abstract — *Time delays are frequently encountered in various areas of science and engineering, including physical and chemical processes, economics, engineering, communication networks and biological systems. The existence of time delays is often a main cause of oscillations, instability and poor performance of the system. During the past decades, the stability analysis of TDS has received considerable attention from researchers, see, e.g. [1, 2] and the references therein. On the other hand, in many practical control systems, the system response is required to be as fast as possible. As a result, the state trajectories of the system are often expected to converge sufficiently fast. Therefore, it is important for designers to be able to estimate the convergence rate of the system. For continuous-time systems with time varying delay, several stability analysis schemes have been proposed for deriving the exponential stability conditions; see., e.g, [3,4,5] and the references therein.*

Keywords — *time-delay system, stability control, Lyapunov function*

I. INTRODUCTION

Along with many advantages of digital control, including cost-effectiveness and high flexibility of embedded systems, the problem of stability analysis for discrete time systems with delay has received considerable attention; see., e.g, [6] and reference therein.

In [7], a delay-dependent stability condition for discrete-time systems with time varying delay was derived by using Moon's inequality, which depends on the minimum and maximum delay bounds. Then, further results were later reported in [8], where a set of augmented LKFs was constructed to use in conjunction with a bounding technique. Another delay-dependent stability criteria of linear continuous/discrete systems with time-varying delay were developed in [9] by using a piecewise analysis method (PAM). Based on the combination of a Lyapunov functional and the delay-partitioning approach, some stability conditions were proposed in [10], where the results were compared with those obtained by using output feedback stabilization in [11]. Improved delay-dependent stabilization criteria were reported in [12] by using a piecewise LKF and a finite sum inequality. By using the model transformation approach, another stability criteria was proposed in terms of linear matrix inequalities [13]. Based on the integral quadratic constraint (IQC) and assumption of bounded interval time-varying delay, a set of new stability criteria was presented in [14]. Recently, some improved results were reported in [13].

However, it should be noted that not much attention has been paid to discrete time systems. Moreover, the conservatism of stability conditions for linear systems with time-varying delay can be relaxed with the combination of existing LKFs, which include quadratic and double summation terms, with some free-weighting matrices. However, the use of FWMs may increase the computational complexity due to an increase in the number of decision variables. As a consequence, it is worth finding a more effective method to ultimately improve stability criteria of these systems that can be obtained in a computationally-effective manner. These together have been the motivation in the current work.

In this chapter, we consider the problem of exponential stability of discrete-time systems with interval time-varying delay. Here, without using any FWMs, we introduce a new set of LKFs containing an augmented vector and some triple summation terms. To enhance the feasible region of stability conditions, the reciprocally convex approach is used to evaluate the double summation terms in the derivative of the proposed LKFs. As a result, improved results on exponential stability are obtained, in comparison with existing stability conditions in the literature. Numerical examples are provided to illustrate the effectiveness of the proposed approach.

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following linear discrete-time system

$$x(k+1) = Ax(k) + Adx(k - \tau(k)), \qquad k \in Z+,$$

$$x(k) = \varphi(k), \qquad k \in Z[-\tau M,0],$$

where $x(k) \in Rn$ is the system state and $A, Ad \in Rn\times n$ are constant matrices. The time-varying delay $\tau(k)$ is assumed to belong to a given interval

$0 < \tau m \leqslant \tau(k) \leqslant \tau M, \forall k \in Z+,$

where $\tau m < \tau M$ are positive constants representing the minimum and maximum delays respectively and $\varphi(k), k \in Z[-\tau M,0]$, is the initial string for system.

It should be noted that system (3.1) is very popular in the literature and it is extensively studied as the time-varying delay τ(k) is frequently encountered in many engineering systems such as networked control systems, chemical process and long transmission lines in pneumatic systems. A typical system containing time delays is the networked control system where the delays induced by the network transmission (either from sensor to controller or from controller to actuator) are actually time-varying.

The aim here is to derive new delay-dependent conditions such that system is exponential stable with the maximum allowable bound for the time delay. The following definition and lemmas are first introduced.

Definition

System is said to be exponentially stable if there exist positive constants α > 1 and N > 1 such that all solutions x(k,φ) of system (3.1) satisfy

$$\|x(k,\phi)\| \le N\|\phi\|\alpha^{-k},\ \forall k \in \mathbb{Z}^{+}$$

where α is the exponential decay rate of system and $\|\phi\| = \max\{\|\phi(k)\| : k \in Z[-\tau M,0]\}$..

For the Lyapunov-Krasovskii method, the construction of LKFs plays a crucial role in deriving the less conservative delay-dependent stability conditions. However, the estimation of the double summation terms in the different LKFs is always a challenging problem. Thus, the following lemma is used frequently.

Lemma1 Let P be a symmetric positive-definite matrix and τ1, τ2 ∈ Z, 0 < τ1 < τ2. Then for any r > 1 and a vector function x(k),k ∈ Z, the following inequality holds

$$a)\ \sum_{s=k-\tau_2}^{k-\tau_1} r^{s-k}x^T(s)Px(s) \ge r_a\left[\sum_{s=k-\tau_2}^{k-\tau_1} x(s)\right]^T P\left[\sum_{s=k-\tau_2}^{k-\tau_1} x(s)\right], \tag{3.3a}$$

$$b)\ \sum_{s=-\tau_2}^{-\tau_1-1}\sum_{u=s}^{-1} r^{u}x^T(k+u)Px(k+u) \ge r_b\left[\sum_{s=-\tau_2}^{-\tau_1-1}\sum_{u=s}^{-1} x(k+u)\right]^T P\left[\sum_{s=-\tau_2}^{-\tau_1-1}\sum_{u=s}^{-1} x(k+u)\right] \tag{3.3b}$$

where

$$r_a = \frac{1-r}{r^{\tau_1}-r^{\tau_2+1}},\ r_b = \frac{(1-r)^2}{r\left[r^{1+\tau_2}-r^{1+\tau_1}+(1-r)(\tau_2-\tau_1)\right]}$$

Proof. By using the Schur complement, we have the following inequality for any

r > 1 and s ∈ Z

$$\begin{bmatrix} r^{s-k}x^T(s)Px(s) & x^T(s) \\ \star & r^{k-s}P^{-1} \end{bmatrix} \ge 0$$

For k − τ2 ⩽ s ⩽ k − τ1, by summing the above inequality from k − τ2 to k − τ1, we Obtain

$$\begin{bmatrix} \sum_{k-\tau_2}^{k-\tau_1} r^{s-k}x^T(s)Px(s) & \sum_{k-\tau_2}^{k-\tau_1} x^T(s) \\ \star & \sum_{k-\tau_2}^{k-\tau_1} r^{k-s}P^{-1} \end{bmatrix} \ge 0$$

By using Schur complement again, it follows that

$$\sum_{k-\tau_2}^{k-\tau_1} r^{-s}x^T(s)Px(s) \ge \left[\sum_{k-\tau_2}^{k-\tau_1} x(s)\right]^T \left(\sum_{k-\tau_2}^{k-\tau_1} r^{k-s}P^{-1}\right)^{-1} \left[\sum_{k-\tau_2}^{k-\tau_1} x(s)\right]$$

With

$$\left(\sum_{k-\tau_2}^{k-\tau_1} r^{k-s}P^{-1}\right)^{-1} = \frac{1-r}{r^{\tau_2}-r^{\tau_1+1}}P = r_aP.$$

It can be seen that the inequality (3.6) is equivalent to the inequality (3.3a). Similarly, for s ⩽ u ⩽ −1 and −τ2 ⩽ s ⩽ −τ1 − 1, by changing the variable s into u in (3.5), we obtain the following inequality

$$\sum_{s=-\tau_2}^{-\tau_1-1} \begin{bmatrix} \sum_s^{-1} r^u x^T(k+u)Px(k+u) & \sum_s^{-1} x^T(k+u) \\ \star & \sum_s^{-1} r^{-u}P^{-1} \end{bmatrix} \ge 0$$

Therefore, the following inequality can be obtained as

$$\sum_{s=-\tau_2}^{-\tau_1-1}\sum_{s}^{-1} r^u x^T(k+u)Px(k+u) \ge \left[\sum_{s=-\tau_2}^{-\tau_1-1}\sum_{s}^{-1} x(k+u)\right]^T \left[\sum_{s=-\tau_2}^{-\tau_1-1}\sum_{s}^{-1} r^{-u}P^{-1}\right]^{-1} \times \left[\sum_{s=-\tau_2}^{-\tau_1-1}\sum_{s}^{-1} x(k+u)\right], \tag{3.8}$$

With

$$\left[\sum_{s=-\tau_2}^{-\tau_1-1}\sum_{s}^{-1} r^{-u}P^{-1}\right]^{-1} = \left[\sum_{s=-\tau_2}^{-\tau_1-1}\sum_{s}^{-1} r^{-u}\right]^{-1} P$$

$$= r_bP.$$

The proof is completed.

The reciprocally convex combination lemma provided in [105] is used in this chapter. This inequality is reformulated as follows:

Lemma2 For a given scalar β ∈ (0,1), an n × n-matrix R > 0 and two

vectors η1, η2 ∈ Rn, define function Θ(β,R) as

$$\Theta(\beta,R) = \frac{1}{\beta}\eta_1^T R\eta_1 + \frac{1}{1-\beta}\eta_2^T R\eta_2$$

If there is a matrix $X \in \mathbb{R}^{n\times n}$ *such that* $\begin{bmatrix} R & X \\ * & R \end{bmatrix} \ge 0$, *then the following inequality holds*

$$\min_{\beta\in(0,1)} \Theta(\beta,R) \ge \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}.$$

Lemma3 Let V (k) be a Lyapunov functional, if there exist scalars γ1 > 0, γ2 > 0 and r > 1 such that

$$\gamma_1\|x(k)\|^2 \le V(k) \le \gamma_2\|x(k)\|^2,$$

$$\Delta V(k) + (1-r^{-1})V(k) \le 0, \quad k \in \mathbb{Z}^{+}$$

then every solution x(k,φ) of system satisfies the following estimation

$$\|x(k,\phi)\| \le \sqrt{\frac{\gamma_2}{\gamma_1}}\|\phi\|\alpha^{-k}, \quad k \in \mathbb{Z}^+$$

where the Lyapunov factor and exponential decay rate are respectively determined as

$N = \sqrt{\frac{\gamma_2}{\gamma_1}}$ and $\alpha = \sqrt{r}$.

Proof we have

V (k + 1)⩽r−1V (k)

⩽...

⩽r−k−1V (0).

Thus

V (k) ⩽ γ2r−k φ 2,k ∈ Z+.

By taking into account, we obtain

$$\|x(k,\phi)\| \le \sqrt{\frac{\gamma_2}{\gamma_1}}\|\phi\|r^{-\frac{1}{2}k} = N\|\phi\|\alpha^{-k}, \quad k \in \mathbb{Z}^+$$

where $\alpha = \sqrt{r}$. This completes the proof.

III. Main Results

The following notations are specifically used in this development. For given integers τ_m,τ_M satisfying $0 < \tau_m < \tau_M$ and any integer number $\delta \in (0,\tau_M-\tau_m)$, matrices X,G and symmetric positive definite matrices $P,Q_j,R_j,j = 1,2,3,S_1,S_2$ of appropriate dimensions, let us denote $\tau = \tau_m+\delta$. We also denote $e_i = [0_{n\times(i-1)n}\ I_n\ 0_{n\times(12-i)n}]$, $i = 1,2,...,12$, as entry matrices, and the following constants.

$$\tau_a = \frac{\tau_m(\tau_m+1)}{2}, \tau_b = \frac{(\tau_M-\tau_m)(\tau_M+\tau_m+1)}{2},$$

$$r_1 = \frac{1-r}{r-r^{\tau_m+1}}, r_2 = \frac{1-r}{r^{\tau+1}-r^{\tau_M+1}}, r_3 = \frac{1-r}{r^{\tau_m+1}-r^{\tau+1}}$$

$$r_4 = \frac{(1-r)^2}{r\left[r^{1+\tau_m}-(1+\tau_m)r+\tau_m\right]},$$

$$r_5 = \frac{(1-r)^2}{r\left[r^{1+\tau_M}-r^{1+\tau_m}+(1-r)(\tau_M-\tau_m)\right]},$$

Vectors

$$\xi(k) = \Big[x^T(k)\ \ x^T(k-\tau_m)\ \ x^T(k-\tau(k))\ \ x(k-\tau)\ \ x^T(k-\tau_M)\ \ \sum_{s=k-\tau_m}^{k-1}x^T(s)\ \ \sum_{s=k-\tau}^{k-\tau(k)-1}x^T(s)\ \ \sum_{s=k-\tau(k)}^{k-\tau_m-1}x^T(s)\ \ \sum_{s=k-\tau_M}^{k-\tau-1}x^T(s)\ \ \sum_{s=k-\tau(k)}^{k-\tau-1}x^T(s)\ \ \sum_{s=k-\tau}^{k-\tau_m-1}x^T(s)\ \ \sum_{s=k-\tau_M}^{k-\tau(k)-1}x^T(s)\Big]^T,$$

$$\zeta(k) = \left[x^T(k)\ \ \sum_{s=k-\tau_m}^{k-1}x^T(s)\ \ \sum_{s=k-\tau}^{k-\tau_m-1}x^T(s)\ \ \sum_{s=k-\tau_M}^{k-\tau-1}x^T(s)\right]^T$$

$$\rho(k) = \left[x^T(k)\ y^T(k)\right]^T,$$

And matrices

Rc = τmR1 +(τM −τ)R2 +(τ −τm)R3, Π1 = [(Ae1 + Ade3)T (e1 −e2 + e6)T (e2 −e4 + e11)T (e4 −e5 + e9)T]T, Π2 =[eT 1 eT 6 eT 11 eT 9]T,Π3 =[eT 1 ((A−I)e1 + Ade3)T]T, Π4 =[eT 6 (e1 −e2)T]T,Π5 =[eT 9 (e4 −e5)T]T,

Π6 =[eT 7 (e3 −e4)T eT 8 (e2 −e3)T]T,Π7 =(A−I)e1 + Ade3,

$$\Pi_8 = \tau_m e_1 - e_6, \Pi_9 = (\tau_M-\tau_m)e_1 - e_8 - e_{12},$$

$$\Pi_{10} = [e_{11}^T\ \ (e_2-e_4)^T]^T, \Pi_{11} = [e_{12}^T\ \ (e_3-e_5)^T\ \ e_{10}^T\ \ (e_4-e_3)^T]^T,$$

$$\Xi_1 = \begin{bmatrix} R_3 & X_1 \\ \star & R_3 \end{bmatrix}; \Xi_2 = \begin{bmatrix} R_2 & X_2 \\ \star & R_2 \end{bmatrix},$$

$$\begin{aligned}\Omega_0(r) = {} & \Pi_1^T P \Pi_1 - r^{-1}\Pi_2^T P \Pi_2 + e_1^T Q_1 e_1 + r^{-\tau_m} e_2^T (Q_2-Q_1) e_2 \\ & + r^{-\tau} e_4^T (Q_3-Q_2) e_4 - r^{-\tau_M} e_5^T Q_3 e_5 + \Pi_3^T R_c \Pi_3 - r_1 \Pi_4^T R_1 \Pi_4 \\ & + \Pi_7^T (\tau_a S_1 + \tau_b S_2) \Pi_7 - r_4 \Pi_8^T S_1 \Pi_8 - r_5 \Pi_9^T S_2 \Pi_9,\end{aligned}$$

$$\Omega_1(r) = r_2 \Pi_5^T R_2 \Pi_5 + r_3 \Pi_6^T \Xi_1 \Pi_6,$$

$$\Omega_2(r) = r_3 \Pi_{10}^T R_3 \Pi_{10} + r_2 \Pi_{11}^T \Xi_2 \Pi_{11},$$

Now, we are ready to present the main result that gives sufficient conditions for exponentially stable of system as follows:

Theorum: For given integers $0 < \tau_m < \tau_M$, if there exist a scalar $r > 1$, an integer $\delta \in (0,\tau_M-\tau_m)$, symmetric positive definite matrices $P,Q_j,R_j,j = 1,2,3,S_1,S_2$ and a matrix X_1,X_2 such that the following inequalities hold

$\Omega_0(r) - \Omega_1(r) < 0$,

$\Omega_0(r) - \Omega_2(r) < 0$

$\Xi_1 \geqslant 0, \Xi_2 \geqslant 0$,

Then system is exponentially stable with the exponential decay rate $\alpha = \sqrt{r}$.

Moreover, every solutions of system satisfies

$$\|x(k,\phi)\| \le \sqrt{\frac{\gamma_2}{\gamma_1}}\|\phi\|\alpha^{-k}, \quad k \in \mathbb{Z}^+$$

Where

$$\gamma_1 = \lambda_m(P),$$

$$\begin{aligned}\gamma_2 = {} & (1+\tau_M)\lambda_M(P) \\ & + \frac{1-r^{-\tau_m}}{1-r^{-1}}\lambda_M(Q_1) + \frac{r^{-\tau_m}-r^{-\tau}}{1-r^{-1}}\lambda_M(Q_2) + \frac{r^{-\tau}-r^{-\tau_M}}{1-r^{-1}}\lambda_M(Q_3) \\ & + 9\frac{(1-r^{-1})\tau_m + r^{-\tau_m-1} - r^{-1}}{(1-r^{-1})^2}\lambda_M(R_1) \\ & + 9\frac{(1-r^{-1})(\tau_M-\tau) + r^{-\tau_M-1} - r^{-\tau-1}}{(1-r^{-1})^2}\lambda_M(R_2) \\ & + 9\frac{(1-r^{-1})(\tau-\tau_m) + r^{-\tau-1} - r^{-\tau_m-1}}{(1-r^{-1})^2}\lambda_M(R_3) \\ & + 4\frac{\tau_a(1-r^{-1})^2 + (1+\tau_m)r^{-2} - \tau_m r^{-1} - r^{-\tau_m-2}}{(1-r^{-1})^3}\lambda_M(S_1) \\ & + 4\frac{\lambda_M(S_2)}{(1-r^{-1})^3}\left[\tau_b(1-r^{-1})^2 + r^{-\tau-2} - (\tau_M-\tau_m)(r^{-1}-r^{-2}) - r^{-\tau_M-2}\right]\end{aligned}$$

Proof: Define $y(k) = x(k+1)-x(k) = (A-I)x(k)+A_dx(k-\tau(k))$. Now, for the sake of getting more information on system the interval time-varying delay $[\tau_m\ \tau_M]$ is divided into two nonuniform subintervals. Note that, for any $k \in Z$, we have either $\tau(k) \in [\tau_m,\tau]$ or $\tau(k) \in (\tau,\tau_M]$. Let us define two sets

$$\Gamma_1 = \{k \in Z | \tau(k) \in [\tau_m,\tau]\}$$

$$\Gamma_2 = \{k \in Z | \tau(k) \in (\tau,\tau_M]\}.$$

Consider the following Lyapunov-Krasovskii functional

$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k)$

Where

$$V_1(k) = \zeta^T(k)P\zeta(k),$$

$$V_2(k) = \sum_{s=k-\tau_m}^{k-1} r^{s-k+1}x^T(s)Q_1x(s) + \sum_{s=k-\tau}^{k-\tau_m-1} r^{s-k+1}x^T(s)Q_2x(s) + \sum_{s=k-\tau_M}^{k-\tau-1} r^{s-k+1}x^T(s)Q_3x(s),$$

$$V_3(k) = \sum_{s=-\tau_m}^{-1}\sum_{v=k+s}^{k-1} r^{v-k+1}\rho^T(v)R_1\rho(v) + \sum_{s=-\tau_M}^{-\tau-1}\sum_{v=k+s}^{k-1} r^{v-k+1}\rho^T(v)R_2\rho(v), + \sum_{s=-\tau}^{-\tau_m-1}\sum_{v=k+s}^{k-1} r^{v-k+1}\rho^T(v)R_3\rho(v),$$

$$V_4(k) = \sum_{s=-\tau_m}^{-1}\sum_{u=s}^{-1}\sum_{v=k+u}^{k-1} r^{v-k+1}y^T(v)S_1y(v) + \sum_{s=-\tau_M}^{-\tau_m-1}\sum_{u=s}^{-1}\sum_{v=k+u}^{k-1} r^{v-k+1}y^T(v)S_2y(v)$$

By taking the forward difference of V1(k) along the solutions of system, we have

$\Delta V1(k) = \zeta T\ (k + 1)P\zeta(k + 1) - r{-}1\zeta T\ (k)P\zeta(k) + (r{-}1 - 1)V1(k),$

Where

$$\zeta(k+1) = \begin{bmatrix} x(k+1) \\ \sum_{s=k+1-\tau_m}^{k} x(s) \\ \sum_{s=k+1-\tau}^{k-\tau_m} x(s) \\ \sum_{s=k+1-\tau_M}^{k-\tau} x(s) \end{bmatrix} = \begin{bmatrix} Ax(k) + A_dx(k-\tau(k)) \\ x(k) - x(k-\tau_m) + \sum_{s=k-\tau_m}^{k-1} x(s) \\ x(k-\tau_m) - x(k-\tau) + \sum_{s=k-\tau}^{k-\tau_m-1} x(s) \\ x(k-\tau) - x(k-\tau_M) + \sum_{s=k-\tau_M}^{k-\tau-1} x(s) \end{bmatrix}$$

Therefore, ΔV1(k) can be obtained of the form

$$\Delta V_1(k) = \xi^T(k)(\Pi_1^T P\Pi_1 - r^{-1}\Pi_2^T P\Pi_2)\xi(k) + (r^{-1}-1)V_1(k)$$

The forward differences of V2(k) and V3(k) are obtained as

ΔV2(k) = xT (k)Q1x(k) + r−τmxT (k − τm)Q2x(k − τm) + r−τxT (k − τ)Q3x(k − τ)

− r−τmxT (k − τm)Q1x(k − τm) − r−τxT (k − τ)Q2x(k − τ)

− r−τMxT (k − τM)Q3x(k − τM) + (r−1 − 1)V2(k),

And

$$\Delta V_3(k) = \rho^T(k)R_c\rho(k) - \sum_{s=k-\tau_m}^{k-1} r^{s-k}\rho^T(s)R_1\rho(s) - \sum_{s=k-\tau_M}^{k-\tau-1} r^{s-k}\rho^T(s)R_2\rho(s) - \sum_{s=k-\tau}^{k-\tau_m-1} r^{s-k}\rho^T(s)R_3\rho(s) + (r^{-1}-1)V_3(k). \tag{3.16}$$

Similarly, the difference of V4(k) along solutions of system is calculated as follows.

$$\Delta V_4 = y^T(k)(\tau_a S_1 + \tau_b S_2)y(k) - \sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1} r^v y^T(k+v)S_1y(k+v) - \sum_{s=-\tau_M}^{-\tau_m-1}\sum_{v=s}^{-1} r^v y^T(k+v)S_2y(k+v) + (r^{-1}-1)V_4(k).$$

We distinguish two sub-intervals of the time delay in the following cases.

Case I: For k ∈ Γ1, i.e. the time delay τ(k) ∈ [τm, τ], from Lemma 1, the following estimations are obtained as

$$-\sum_{s=k-\tau_m}^{k-1} r^{s-k}\rho^T(s)R_1\rho(s) \le -r_1\begin{bmatrix} \sum_{s=k-\tau_m}^{k-1} x(s) \\ x(k) - x(k-\tau_m) \end{bmatrix}^T R_1 \begin{bmatrix} \sum_{s=k-\tau_m}^{k-1} x(s) \\ x(k) - x(k-\tau_m) \end{bmatrix} \le -r_1\xi^T(k)\Pi_4^T R_1\Pi_4\xi(k).$$

And

$$-\sum_{s=k-\tau_M}^{k-\tau-1} r^{s-k}\rho^T(s)R_2\rho(s) \le -r_2\begin{bmatrix} \sum_{s=k-\tau_M}^{k-\tau-1} x(s) \\ x(k-\tau) - x(k-\tau_M) \end{bmatrix}^T R_2 \times \begin{bmatrix} \sum_{s=k-\tau_M}^{k-\tau-1} x(s) \\ x(k-\tau) - x(k-\tau_M) \end{bmatrix} \le -r_2\xi^T(k)\Pi_5^T R_2\Pi_5\xi(k).$$

More we also have

$$-\sum_{s=k-\tau}^{k-\tau_m-1} r^{s-k}\rho^T(s)R_3\rho(s) = -\sum_{s=k-\tau}^{k-\tau(k)-1} r^{s-k}\rho^T(s)R_3\rho(s) - \sum_{s=k-\tau(k)}^{k-\tau_m-1} r^{s-k}\rho^T(s)R_3\rho(s)$$

By using

Lemma 1, the following inequality is obtained as

$$-\sum_{s=k-\tau}^{k-\tau(k)-1} r^{s-k}\rho^T(s)R_3\rho(s) - \sum_{s=k-\tau(k)}^{k-\tau_m-1} r^{s-k}\rho^T(s)R_3\rho(s) \le -\frac{1-r}{r^{\tau(k)+1}-r^{\tau+1}}\Big[\sum_{s=k-\tau}^{k-\tau(k)-1}\rho(s)\Big]^T R_3\Big[\sum_{s=k-\tau}^{k-\tau(k)-1}\rho(s)\Big] - \frac{1-r}{r^{\tau_m+1}-r^{\tau(k)+1}}\Big[\sum_{s=k-\tau(k)}^{k-\tau_m-1}\rho(s)\Big]^T R_3\Big[\sum_{s=k-\tau(k)}^{k-\tau_m-1}\rho(s)\Big]$$

From lemma2, we have

$$-\sum_{s=k-\tau}^{k-\tau(k)-1} r^{s-k}\rho^T(s)R_3\rho(s) - \sum_{s=k-\tau(k)}^{k-\tau_m-1} r^{s-k}\rho^T(s)R_3\rho(s) \le -r_3\begin{bmatrix}\eta_1 \\ \eta_2\end{bmatrix}^T \begin{bmatrix} R_3 & X_1 \\ \star & R_3 \end{bmatrix}\begin{bmatrix}\eta_1 \\ \eta_2\end{bmatrix} \le -r_3\xi^T(k)\Pi_6^T\Xi_1\Pi_6\xi(k),$$

Where $\beta_1 = \dfrac{r^{\tau(k)} - r}{r^{\tau_m} - r^{\tau}}$ and

$$\eta_1 = \begin{bmatrix} \sum_{s=k-\tau}^{k-\tau(k)-1} x(s) \\ x(k-\tau(k)) - x(k-\tau) \end{bmatrix}, \eta_2 = \begin{bmatrix} \sum_{s=k-\tau(k)}^{k-\tau_m-1} x(s) \\ x(k-\tau_m) - x(k-\tau(k)) \end{bmatrix}$$

Similarly, for the different ΔV4(k), by using Lemma 1, the following inequalities are obtained as

$$-\sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1} r^v y^T(k+v)S_1y(k+v) \le -r_4\Big(\sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1} y(k+v)\Big)^T S_1\Big(\sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1} y(k+v)\Big) \le -r_4\Big(\tau_m x(k) - \sum_{s=k-\tau_m}^{k-1} x(s)\Big)^T S_1\Big(\tau_m x(k) - \sum_{s=k-\tau_m}^{k-1} x(s)\Big) \le -r_4\xi^T(k)\Pi_8^T S_1\Pi_8\xi(k).$$

And

$$
\begin{aligned}
&-\sum_{s=-\tau_M}^{-\tau_m-1}\sum_{v=s}^{-1} r^v y^T(k+v)S_2 y(k+v)\\
&\le -r_5\left(\sum_{s=-\tau_M}^{-\tau_m-1}\sum_{v=s}^{-1} y(k+v)\right)^T S_2\left(\sum_{s=-\tau_M}^{-\tau_m-1}\sum_{v=s}^{-1} y(k+v)\right)\\
&\le -r_5\left[(\tau_M-\tau_m)x(k)-\sum_{s=k-\tau}^{k-\tau_m-1}x(s)\right]^T S_2\left[(\tau_M-\tau_m)x(k)-\sum_{s=k-\tau_M}^{k-\tau_m-1}x(s)\right] \qquad (3.24)\\
&\le -r_5\left[(\tau_M-\tau_m)x(k)-\sum_{s=k-\tau_M}^{k-\tau(k)-1}x(s)-\sum_{s=k-\tau(k)}^{k-\tau_m-1}x(s)\right]^T S_2\\
&\quad\times\left[(\tau_M-\tau_m)x(k)-\sum_{s=k-\tau}^{k-\tau(k)-1}x(s)-\sum_{s=k-\tau(k)}^{k-\tau_m-1}x(s)\right]\\
&\le -r_5\xi^T(k)\Pi_9^T S_2\Pi_9\xi(k).
\end{aligned}
$$

We now obtain

$\Delta V(k) + (1 - r^{-1})V(k) \leqslant \xi^T(k)(\Omega_0 - \Omega_1)\xi(k)$, $\forall k \in Z+$. (3.25) Case II: For $k \in \Gamma_2$, i.e. the time delay $\tau(k) \in (\tau,\tau_M]$, by Lemma 1, we obtain

$$
\begin{aligned}
&-\sum_{s=k-\tau}^{k-\tau_m-1} r^{s-k}\rho^T(s)R_3\rho(s)\\
&\le -r_3\begin{bmatrix}\sum_{s=k-\tau}^{k-\tau_m-1}x(s)\\ x(k-\tau_m)-x(k-\tau)\end{bmatrix}^T R_3\begin{bmatrix}\sum_{s=k-\tau}^{k-\tau_m-1}x(s)\\ x(k-\tau_m)-x(k-\tau)\end{bmatrix}\\
&\le -r_3\xi^T(k)\Pi_{10}^T R_3\Pi_{10}\xi(k).
\end{aligned}
$$

In addition, we have

$$
-\sum_{s=k-\tau_M}^{k-\tau-1} r^{s-k}\rho^T(s)R_2\rho(s) = -\sum_{s=k-\tau_M}^{k-\tau(k)-1} r^{s-k}\rho^T(s)R_2\rho(s) - \sum_{s=k-\tau(k)}^{k-\tau-1} r^{s-k}\rho^T(s)R_2\rho(s)
$$

Similarly, we have the following estimation

$$
\begin{aligned}
&-\sum_{s=k-\tau_M}^{k-\tau(k)-1} r^{s-k}\rho^T(s)R_2\rho(s) - \sum_{s=k-\tau(k)}^{k-\tau-1} r^{s-k}\rho^T(s)R_2\rho(s)\\
&\le -\frac{1-r}{r^{\tau(k)+1}-r^{\tau_M+1}}\left[\sum_{s=k-\tau_M}^{k-\tau(k)-1}\rho(s)\right]^T R_2\left[\sum_{s=k-\tau_M}^{k-\tau(k)-1}\rho(s)\right]\\
&-\frac{1-r}{r^{\tau+1}-r^{\tau(k)+1}}\left[\sum_{s=k-\tau(k)}^{k-\tau-1}\rho(s)\right]^T R_2\left[\sum_{s=k-\tau(k)}^{k-\tau-1}\rho(s)\right],
\end{aligned}
$$

Which yields

$$
\begin{aligned}
&-\sum_{s=k-\tau_M}^{k-\tau(k)-1} r^{s-k}\rho^T(s)R_2\rho(s) - \sum_{s=k-\tau(k)}^{k-\tau-1} r^{s-k}\rho^T(s)R_2\rho(s)\\
&\le -r_2\begin{bmatrix}\eta_3\\ \eta_4\end{bmatrix}^T\begin{bmatrix}R_2 & X_2\\ \star & R_2\end{bmatrix}\begin{bmatrix}\eta_3\\ \eta_4\end{bmatrix}\\
&\le -r_2\xi^T(k)\Pi_{11}^T\Xi_2\Pi_{11}\xi(k),
\end{aligned}
$$

M

Where $\beta_2 = \dfrac{r^{\tau(k)} - r^{\tau}}{r^{\tau} - r^{\tau_M}}$ and

$$
\eta_3 = \begin{bmatrix}\sum_{s=k-\tau_M}^{k-\tau(k)-1}x(s)\\ x(k-\tau(k))-x(k-\tau_M)\end{bmatrix}, \eta_4 = \begin{bmatrix}\sum_{s=k-\tau(k)}^{k-\tau-1}x(s)\\ x(k-\tau)-x(k-\tau(k))\end{bmatrix}
$$

Similarly, we obtain the following inequality

$$
\Delta V(k) + (1 - r^{-1})V(k) \le \xi^T(k)(\Omega_0 - \Omega_2)\xi(k), \quad \forall k \in Z^+.
$$

It follows from Lemma 3

$\Delta V(k) + (1 - r^{-1})V(k) \leqslant 0, \qquad \forall k \in Z+$.

On the other hand, it can be verified from (3.13) that

$$
\|x(k,\phi)\| \le N\|\phi\|\alpha^{-k}, \quad k \in \mathbb{Z}^+
$$

where $N = \sqrt{\dfrac{\gamma_2}{\gamma_1}}$ Thus, from Definition 3, system is exponentially stable with the exponential decay rate $\alpha = \sqrt{r}$. The proof is completed.

Remark 1 By comparison to other LKFs existing in the literature, the new LKFs proposed in this paper contain a new augmented vector $\zeta(k)$ in $V_1(k)$ and two triple summation terms in $V_4(k)$. As a result, the information about the current values of the state variables $x(k)$ and their history are exploited, leading to less conservative stability conditions.

Remark 2 It should be noted that the stability conditions, established in Theorem 20, contain the tuning parameters α and δ so there remains the interesting question as how to find the optimal combination of these parameters. A direct method to solve that problem is to choose a cost function tmin that is obtained while solving the feasibility problem using Matlab's LMI toolbox. When tmin is positive, the combination of the tuning parameters α and δ does not allow a feasible solution to the set of LMIs [15]. By using a numerical optimisation algorithm, such as the program fminsearch in the optimisation toolbox of Matlab, we can find the solution of the cost function tmin. The optimal combination of two parameters is obtained when the minimum value of the cost function is negative [16].

Remark 3 The convergence rate of the system can be directly chosen from this proposed approach. Moreover, the exponential stability conditions given in Theorem 20 are derived in terms of LMIs without introducing any free-weighting matrices. Therefore, the obtained stability conditions may involve fewer decision variables, and hence reduce the computational complexity.

IV. Numerical examples

For illustration of the effectiveness of the proposed approach in relaxing the conservatism of the stability conditions, let us consider the system, given in [7, 11, 14, 10] as follows

$$
x(k+1) = \begin{pmatrix}0.8 & 0\\ 0.05 & 0.9\end{pmatrix}x(k) + \begin{pmatrix}-0.1 & 0\\ -0.2 & -0.1\end{pmatrix}x(k-\tau(k))
$$

For this system, the asymptotic stability conditions proposed in the aforementioned studies gave the allowable upper bounds τ_M with various values of τ_m as listed in Table 3.1 below. Here, from Theorem 20 and Remark 2, we find the optimal value for r as r

= 1.001. As a result, the exponential convergence rate is calculated as α = 1.0005 and the upper bounds τM are also given correspondingly in Table 3.1. It can be seen that all the upper bounds τM obtained by using Theorem 20 are larger than those obtained in the papers mentioned above. As explained in Remark 1, this also confirms the improvement of this approach on stability conditions as compared to existing results.

TABLE I

MABS OF TM FOR DIFFERENT VALUES OF TM

τm	1	3	5	7	11	13	15
Gao et al. (2007)	12	13	13	14	16	17	18
Zhang et al. (2008)	12	13	14	15	17	19	20
He et al. (2008)	17	17	18	18	20	22	23
Meng et al. (2010)	17	17	18	18	20	22	23
Li and Gao (2011)	-	18	19	21	25	25	26
Kao (2012)	17	18	19	21	25	25	-
This paper	19	19	20	21	22	23	24

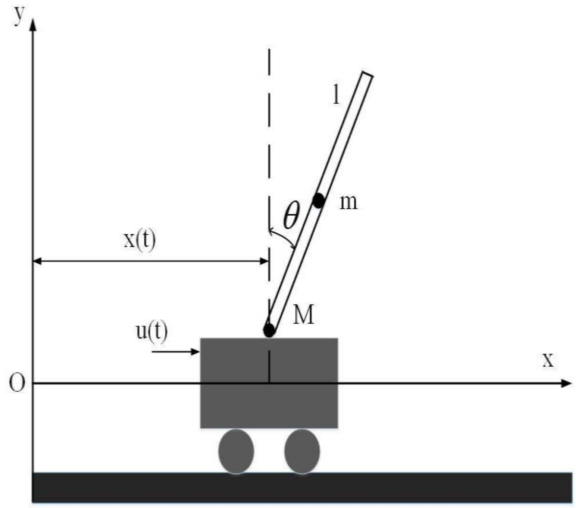

Figure Inverted Pendulum

V. Conclusion

This chapter has addressed the problem of exponential stability for a class of discrete-time systems with interval time-varying delay. A new set of Lyapunov-

Krasvoskii functionals containing an augmented vector and some triple summations

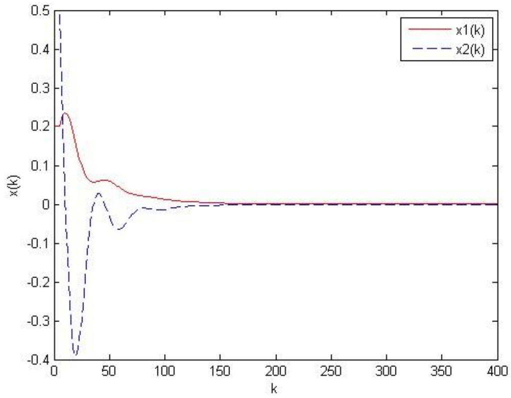

Figure The state trajectories of x1(k) and x2(k).

has been proposed. By combining the reciprocally convex approach with the delaydecomposition technique, improved exponential stability conditions are derived in terms of LMIs. In comparison with existing results in the literature, the obtained conditions are less conservative, judging by a larger maximum allowable bound. The effectiveness of the proposed approach is illustrated through numerical examples.

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