Computing the Solution of the Hartree Equation with Repulsive Harmonic Potential

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***Abstract*—**We study the computability of the solution operator of the initial problem for the Hartree equation with repulsive harmonic potential on the Type-2 Turing machines. We will prove that in Sobolev space $\sum = H^1 \cap FH^1$, for $n \geq 5$,when the solution operator: $K_R : \Sigma(R^n) \to C(R; \Sigma(R^n))$ is $(\delta_{H^s}, [\rho - \delta_{H^s}])$ -computable. The conclusion enriches the theory of computability.

***Keywords*—Hartree equation with repulsive harmonic potential, TTE, Sobolev space, Initial problem**

I. Introduction

At present, the computability of solutions of the nonlinear evolution equations have become an important topic to the workers of physics and mathematics. Researching boundedness and computability of the solutions of the nonlinear equations will offer effective tools for the application of equations, enrich theoretical foundation of computer science and promote the development of computer software. From 1985，K.Weihrauch and others established a computational model, called Type-2 theory of effectivity (TTE for short). K.Weihrauch and N.Zhong have studied the computability of the solution operator of a three-dimensional wave equation :

$$u_{tt} = u, u(0,x) = f(x), u_t(0,x) = 0, t \in R, x \in R^3$$

on Sobolev space by using Type-2 theory of Effectivity, and construct the appropriate space to prove its unique solution is computable in the scope of the continuous differential equation. Dianchen Lu and others have studied the computability of the non-linear kawahara equation of[1].

The Hartree equation with repulsive harmonic potential[2-5]:

$$iu_t + \frac{1}{2}\Delta u + \frac{1}{2}|x|^2 u = f(u), \quad x \in R^n \times R, n \geq 5 , \tag{1.1}$$

$$u(0) = \varphi(x), \qquad x \in R^n, \tag{1.2}$$

Here $f(u) = (V * |u|^2)u$ is a nonlinear function of Hartree for $V(x) = |x|^{-\gamma}, \gamma = 4$,where $*$ denotes the convolution in R^n .

In this paper , we will prove that the solution operator of the initial problem(1.1) and (1.2) is computability.

We can get its equivalent integral equation by Duhamel principle:

$$u(t) = U(t) - i\int_0^t U(t-s) f(u(s))ds \tag{1.3}$$

Where $U(t) = e^{\frac{1}{2}it(\Delta + |x|^2)}$.

The structure of the article is that: In part 2, we mainly introduce some basic definitions, lemmas and conclusions, which are relevant to the proof of part3; In part 3, we prove the main theorem of the paper mainly.

II. Preliminaries

Lemma 2.1[6] (1) In Schwarz space $S(R)$,the function $(a,\varphi) \mapsto a\varphi$ is $(\rho, \delta_s, \delta_s)$ -computable; $(\varphi, t) \mapsto \varphi(t)$ is (δ_s, ρ, ρ) - computable; $(\varphi, \phi) \mapsto \varphi + \phi$ is $(\delta_s, \delta_s, \delta_s)$ -computable.

(2)The function $(\varphi, t) \mapsto V(t)\varphi$ is $(\delta_s, \rho, \delta_s)$ -computable.

(3)The fourier transform and its inverse fourier transform are both computable.

Lemma 2.2[6] **(type conversion)** Let $\delta_i :\subseteq \Sigma^\omega \to X_i$ be a representation of the set X_i $(0 \leq i \leq k)$ .let

$$L(x_1, \cdots, x_{k-1})(x_k) := f(x_1, \cdots, x_k),$$

then if f is $(\delta_1, \cdots, \delta_k, \delta_0)$ -computable if and only if L is $(\delta_1, \cdots, \delta_{k-1}, [\delta_k \to \delta_0])$ -computable .

Lemma 2.3[6] The fuction

$$H : C(R; S(R)) \times R \times R \to S(R)$$

$$H(u,a,b) = \int_a^b u(t)dt$$

is $([\rho \to \delta_s], \rho, \rho, \delta_s)$ -computable.

Lemma 2.4[6] Let $\gamma :\subseteq Y \to M$ and $\gamma' :\subseteq Y \to M'$ are two representations, v_N is admissible representation of N .Then we have the following propositions:

(1) If $f :\subseteq M \to M'$ is (γ, γ')- computable, then

$f' :\subseteq N \times M' \times M \to M'$ is $(v_N, \gamma', \gamma, \gamma')$ - computable.

We define a function $g' :\subseteq N \times M \to M'$ as follow:

$$g'(0,x) = f(x), g'(n+1,x) = f'(n, g'(n,x), x),$$

Where $x \in M$, $n \in N$,then g' is (v_N, γ, γ') - computable.

(2)Assuming that $h :\subseteq M \to M$ is (γ, γ) -computable,

Define a function

$$H :\subseteq N \times M \to M :$$

$$H(0,x) = x, H(n+1,x) = h \circ H(n,x) = h^{n+1}(x),$$

So, the function H is (v_N, γ, γ) - computable.

Definition 2.5[7] For any time interval I , we use $L_t^q L_x^r (I \times R^n)$ to denote the mixer space-time Lebesgue norm

$$\|u\|_{L_t^q L_x^r (I \times R^n)} = (\int_I \|u\|_{L^r(R^n)}^q dt)^{\frac{1}{q}}$$

with the usual modifications when $q = \infty$. When $q = r$, we abbreviate $L_t^q L_x^r$ by $L_{t,x}^q$.

For a space-time slab $I \times R^n$,we define the Strichartz norm $\dot{S}^0(I)$ by

$$\|u\|_{\dot{S}^0(I)} = \sup_{(q,r) admissible} \|u\|_{L_t^q L_x^r (I \times R^n)}$$

When $n \geq 5$, the space $(\dot{S}^0(I), \|\cdot\|_{\dot{S}^0(I)})$ is Banach space.

For sake of convenience, we introduce three abbreviated notations. For a time interval I ,we Set

$$X_1(I) = L_t^6 L_x^{\frac{6n}{3n-8}} (I \times R^n) ,$$

$$X_0(I) = L_t^6 L_x^{\frac{6n}{3n-2}} (I \times R^n) ,$$

$$Z_0(I) = L_t^3 L_x^{\frac{6n}{3n-4}} (I \times R^n) .$$

We denote by $A(t)$ $(t \in R)$ the fundamental solution operator:

$$A(t) = \{J(t), H(t), I\} , B = \{i\nabla, x, I\} .$$

$$J(t) = i\nabla \cosh t - x \sinh t , H(t) = -i\nabla \sinh t + x \cosh t .$$

$$\Sigma = \{u : \|u\|_{L^2} + \|\nabla u\|_{L^2} + \|xu\|_{L^2} < \infty\}$$

Lemma 2.6[7] For any function u on $I \times R^n$,we have

$$\|A(t)u(t)\|_2 \leq C(\|u_0\|_\Sigma) \qquad (2.1)$$

Lemma 2.7[7] Let $f(u) = (V * |u|^2)u$, where $V(x) = |x|^{-4}$.

For any time interval I and $t_0 \in I$, we have

$$\left\| \int_0^t A(t)U(t-s) f(u)(s,x) ds \right\|_{\dot{S}^0(I)} \leq \|u\|_{X_1(I)}^2 \|A(t)u\|_{Z_0(I)} \qquad (2.2)$$

Ⅲ. MAIN RESULT

From the problem (1.1) and (1.2), we establish a nonlinear map

$$K_R : \Sigma(R^n) \to C(R; \Sigma(R^n))$$

which translatea the initial data $\varphi \in \Sigma$ to the solution $(0 \leq t \leq T, 0 \leq t' \leq \overline{T})$.The map K_R is the solution operator of the initial problem .

Theorem 3.1 When, $n \geq 5$, $\Sigma = H^1 \cap FH^1$, the solution operator $K_R : \Sigma(R^n) \to C(R; \Sigma(R^n))$ is $(\delta_\Sigma, [\rho - \delta_\Sigma])$ - computable.

To prove the Theorem 3.1, we firstly translate the differential equation to its equivalent integral equation by Duhamel principle on space Σ; Then prove the existence and uniqueness of solution by the contraction principle. Last, using the type-2 Turing machine and some propositions of Sobolev space to prove the solution operator is computable.

We can get its equivalent integral equation by Duhamel principle:

$$u(t) = U(t) - i\int_0^t U(t-s) f(u(s)) ds$$

Where $U(t) = e^{\frac{1}{2}it(\Delta + |x|^2)}$

Now, we prove the existence and uniqueness of solution by the contraction principle, i.e., lemma 3.2.

Lemma 3.2 Let $n \geq 5$, $\gamma = 4$. Then there exist a $T > 0$ and a unique solution u of (1.1) in $u \in C(R; \Sigma(R^n))$

Proof Define the work space as

$$\mathrm{B} = \left\{ u : \|J(t)u\|_{x_0 \cap z_0(I)} \leq 2\eta, \|H(t)u\|_{z_0(I)} \leq 2C\|xu\|_2 , \|u\|_{z_0(I)} \leq 2C\|u_0\|_2 \right\}$$

with the natural metric.

We define an operator Φ :

$$A(t)\Phi(u)(t) := U(t)Bu(t_0) - i\int_0^t U(t-s)A(s) f(u(s)) ds \qquad (3.1)$$

For any $u \in \mathrm{B}$, by Lemmas 2.6- 2.7 we have

$$\|\Phi(u)\|_{z_0(I)} \leq \|u_0\|_2 + C\|u\|_{X_1(I)}^2 \|u\|_{z_0(I)}$$

$$\le \|\varphi\|_2 + C\|u\|^2_{X_1(I)}\|u\|_{z_0(I)}$$
$$\le \|\varphi\|_2 + C\eta^2\|u\|_{z_0(I)}$$
$$\le 2C\|\varphi\|_2 \qquad (3.2)$$

Similarly, we also have

$$\|\Phi(u)-\Phi(v)\|_{z_0(I)} \le (\|u\|^2_{X_1(I)}+\|v\|^2_{X_1(I)})\|u-v\|_{z_0(I)}$$
$$\le (\|J(t)u\|^2_{X_0(I)}+\|J(t)v\|^2_{X_0(I)})\|u-v\|_{z_0(I)}$$
$$\le 2\eta^2\|u-v\|_{z_0(I)} \qquad (3.3)$$

as long as η is chosen sufficiently small. According the Banach fixed point theorem, Φ has the unique fixed point. The point is the solution of the initial problem (1.1) and (1.2). This completes the proof.

For $\varphi \in C(R;\Sigma(R^n))$,define solution operator：

$$S(t)=U(t)Bu(t_0)-i\int_0^t U(t-s)A(s)f(u(s))ds\,.$$

According the Lemma 3.2 in[2], It is easy to prove the operator is $(\delta_\Sigma,[\rho-\delta_\Sigma])$ computable.

Corollary 3.3 Function

$$\bar{S}:C\left(R;S\left(R^2\right)\right)\times S\left(R^2\right)\to C\left(R;S\left(R^2\right)\right)$$
$$\bar{S}\left(u,\phi\right)\left(t\right):=S\left(u,\phi,t\right),$$

is $\left(\delta_s,\nu_N,[\rho\to\delta_s]\right)$-computable.

Proof This follows from lemma 2.2 and lemma 3.2 in [2] .

Lemma 3.4: The function

$v:S(R^2)\times N\to C(R;S(R^2))$,defined by

$$v(\phi,0)=\bar{S}\left(0,\phi\right) \qquad (3.4)$$
$$v(\phi,j+1)=\bar{S}\left(v(\phi,j),\phi\right) \qquad (3.5)$$

is $\left(\delta_\Sigma,[\rho-\delta_\Sigma]\right)$-computable

Proof The function v is defined by primitive recursion fromcomputable functions. By Lemma 2.4 v is $\left(\delta_s,\nu_N,[\rho\to\delta_s]\right)$-computable.

Next, we prove the theorem (3.1).

For given the initial value $\varphi\in C(R;\Sigma(R^n))$ and rational value $T>0$,we will consider the following problem :

$$\begin{cases} iw_t+\frac{1}{2}\Delta w+\frac{1}{2}|x|^2 w = f(w), & x\in R^n\times R, n\ge 5 \\ w(0)=\varphi(x), & x\in R^n, \end{cases} \qquad (3.6)$$

where $w(x,t)=u(x,t+t_0)$, $t_0\ge 0$, $w(x,y,t_0)=\varphi(x,y)$

We assume that the initial value $\varphi\in C(R;\Sigma(R^n))$ is given by a $\tilde{\delta}_{H^s}$ -name , i.e., $p=\langle p_0,p_1,\cdots\rangle$ which is obtained by $\delta_\Sigma(p_i)=\varphi_i$ and $\|\varphi_n-\varphi\|_{z_0(I)}\le 2^{-n-2}$ $n\in N$. For $\forall k\in N$, there exist appropriate computable n_k satisfying

$\|\varphi_{n_k}-\varphi\|_{z_0(I)}\le 2^{-n_k-2}\le 2^{-k-2}$.

Define

$$w_n^0:=\bar{S}(0,\varphi_n),\quad w_n^{j+1}:=\bar{S}(w_n^j,\varphi_n)$$

From Lemma (3.2), we know the sequence $\{w_n^j\}$ is computable. If $w_n^j\to w_n(j\to\infty)$, then v_n is the fixed point of the iteration and satisfies the following integral equation:

$$w_n(t)=\bar{S}(w_n,\varphi_n)$$
$$=U(t)Bw_n(t_0)-i\int_0^t U(t-s)A(s)f(w_n(s))ds$$

So, $w_n(t)$ is the solution of the initial problem

$$\begin{cases} i\frac{\partial w_n}{\partial t}+\frac{1}{2}\Delta w_n+\frac{1}{2}|x|^2 w_n = f(w_n), & x\in R^n\times R, n\ge 5 \\ w_n(0)=\varphi(x), & x\in R^n, \end{cases} \qquad (3.7)$$

Since $w_n^j\to w_n(j\to\infty)$,we can select suitable integer n_k , j_k to constrct a sequence $\{w_{n_k}^{j_k}\}_{k\in N}$,satisfying $\|w_{n_k}^{j_k}-w_{n_k}\|_{z_0(I)}\le 2^{-k-1}$ .Then $\{w_{n_k}^{j_k}\}_{k\in N}$ is computable sequence. For n_k , j_k and $\{w_n^j\}$ are computable, the δ_Σ – name of $w_{n_k}^{j_k}(t)$ is compute by Lemma 2.1.(1).

In the following ,we prove $\{w_{n_k}^{j_k}\}_{k\in N}$ fastly converges to w .

From lemma (2.6)-(2.7),

$$\|w_{n_k}-w\|_{z_0(I)}\le\|\varphi_{n_k}-\varphi\|_{z_0(I)}+(\|w_{n_k}\|^2_{X_1(I)}+\|w\|^2_{X_1(I)})\|w_{n_k}-w\|_{z_0(I)}$$
$$\le 2^{-k-2}+2\eta^2\|w_{n_k}-w\|_{z_0(I)}$$

When T sufficient small such that $0 < \frac{1}{1-2\eta^2} < 2$, then

$$\left\| w_{n_k} - w \right\|_{z_0(I)} \le 2^{-k-1}$$

Therefore,

$$\left\| w_{n_k}^{j_k} - w \right\|_{z_0(I)} \le \left\| w_{n_k}^{j_k} - w_{n_k} \right\|_{z_0(I)} + \left\| w_{n_k} - w \right\|_{z_0(I)}$$
$$\le 2^{-k-1} + 2^{-k-1} = 2^{-k}$$

Then we have proved $\left\{ w_{n_k}^{j_k} \right\}_{k\in N}$ fastly converges to w and w is computable.

We known $\left\{ w_{n_k}^{j_k} \right\}_{k\in N}$ is computable sequence,if $\delta_\Sigma(q_k) = w_{n_k}^{j_k}(t)$, then $\tilde{\delta}_{z_0(I)} \langle q_0, q_1, \cdots \rangle = v(t)$,i.e., $\langle q_0, q_1, \ldots \rangle$ is the $\tilde{\delta}_{z_0(I)}$-name of $w(t)$. Hence the solution v of the initial problem （3.6） is computable on $t \in [-T, T]$,that is solution operator map S is computable.

We define a $(\delta_\Sigma, [\rho - \delta_\Sigma])$-computable map

$$P : (t_0, \varphi, t) \to u(t), t \in [t_0 - T, t_0 + T],$$

Where $w(t_0) = \varphi$, $v(x)$ is the solution of the initial problem (1.1)and（1.2） on $t \in [t_0, t_0 + T]$.

Then we prove the solution $u(n \cdot T)$ is computable. The function H : $H\ (\varphi, n) = u(nT)$ defined by

$$H\ (\varphi, 0) = \varphi$$
$$H\ (\varphi, n+1) = P\ (nT, H(\varphi, n), (n+1)T)$$

is computable since H is derived by primitive recursion from computable function P.

In the end, we prove $u(t)$ is computable. let $n \cdot T \le t \le (n+1) \cdot T$,we first compute $u(n \cdot T)$,then compute $P\ (nT, u(nT), t)$,so $u(t) = P\ (nT, u(nT), t)$ is computable.

In this way, we have get the computable solution on on $t \in R$. When, $n \ge 5$, $\Sigma = H^1 \cap FH^1$, the solution operator $K_R : \Sigma(R^n) \to C(R; \Sigma(R^n))$ is $(\delta_\Sigma, [\rho - \delta_\Sigma])$-computable.

SUMMARY AND OUTLOOK

The paper study computable of the solution operator of the dissipation-modified Kadomtsev-petviashvili equation .On the basis of computability theory, whether problem can be implemented on computer is an important problem. Computational complexity theory just can be used to solve the problem. The topic we will study in the future.

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